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Wavelets and Pre-wavelets in Low Dimensions*

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In Riemenschneider and Shen (*in* "Approximation Theory and Functional Analysis" (C. K. Chui, Ed.), pp. 133–149, Academic Press, New York, 1991) an explicit orthonormal basis of wavelets for $L^2(\mathbb{R}^s)$, $s = 1, 2, 3$, was constructed from a multiresolution approximation given by box splines. In other words, $L^2(\mathbb{R}^s)$ has the orthogonal decomposition

$$\bigoplus_{v \in \mathbb{Z}} W_v. \tag{$*$}$$

Orthonormal bases for the spaces W_v are given by $\{2^{vs/2} K_\mu(2^v \cdot -j)\}$, $j \in \mathbb{Z}^s$, $\mu \in \mathbb{Z}_2^s \backslash 0$, where $\mathbb{Z}_2^s := \{0, 1\}^s$ and the "wavelets" K_μ are $2^s - 1$ cardinal splines with exponential decay. In this paper, we consider multiresolutions generated by suitable compactly supported and symmetric functions φ and explicitly construct $2^s - 1$ compactly supported functions φ_μ, $\mu \in \mathbb{Z}_2^s \backslash 0$, such that the translates $\varphi_\mu(\cdot - j), j \in \mathbb{Z}^s$, are an unconditional basis for W_0. Thus, the functions $\varphi_\mu(2^v \cdot -j)$, $v \in Z$, $j \in \mathbb{Z}^s$, $\mu \in \mathbb{Z}_2^s \backslash 0$ comprise a basis for the orthogonal decomposition (*) (the functions are orthogonal for different v because the decomposition is orthogonal, but neither the translates nor the functions will be orthogonal for given v). The functions are given as $\varphi_\mu(\cdot/2)/2^s = \varphi *' \beta_\mu$ with the sequences β_μ formed from a single sequence by translation and change in sign pattern. We also discuss various ways to regain some of the orthogonality lost by requiring compact support. © 1992 Academic Press, Inc.

1. INTRODUCTION

In [RS] an orthonormal basis of wavelets was explicitly constructed from a multiresolution approximation for $L^2(\mathbb{R}^s)$, $s = 1, 2, 3$, generated by box splines. For $s = 1$, the box splines are the cardinal splines of Schoen-

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berg and serve as examples of orthogonal wavelets as discussed by Battle [B], Daubechies [D], Lemarie [L], Mallet [Ma], Meyer [M] and others. The novelty of [RS] was that the construction for $s=2, 3$ was not of tensor product type. More generally, the construction in [RS] is valid for any centrally skew-symmetric complex-valued function φ that generates an appropriate multiresolution approximation. For arbitrary dimensions, Jia and Micchelli [JM_2] provided a rather complete analysis of the multiresolution associated with functions satisfying a refinement equation. They gave a general and natural framework for the analysis and gave a general solution for the existence of pre-wavelets. Their solution is based on the notion of extensibility of a finite set of Laurent polynomials and makes use of the Quillen–Suslin Theorem (this approach was used by Micchelli in [Mi]). The method was demonstrated by the construction of compactly supported pre-wavelets for the linear three-direction box spline in $\mathbb{R}^2$ (also see Section 5). As a concrete method of extensibility (for dimensions 1, 2, and 3) in the construction of wavelets, the construction from [RS] was reviewed in [JM_2, Sect. 7] for real symmetric functions and their theory superceded some of the early work on this paper. The main purpose of this note is to provide a concrete way of constructing compactly supported pre-wavelets based on the method in [RS] when the complex-valued function φ that generates the multiresolution approximation is centrally skew-symmetric, satisfies a refinement equation with finite mask, and has l^2-stable integer translates. This gives us the opportunity to recall the construction of wavelets from [RS] in the general case of centrally skew-symmetric functions in Section 2 in order to set the stage for the construction of the compactly supported prewavelets in low dimensions in Section 3. Our search for a convenient construction of compactly supported pre-wavelets was motivated by requests from colleagues for use in applications and the work of Chui and Wang [CW_1, CW_2] in the univariate case. As we were completing this manuscript, we learned that Chui, Stöckler, and Ward [CSW] have independently found a similar construction of pre-wavelets in the case of box splines. Although both methods are based on [RS], they differ in some details and in the surrounding analysis. The remainder of this section will be devoted to recalling the details of the terminology that we have been using.

We put ourselves in the setting proposed by Jia and Micchelli who considered refinable and l^p-stable functions $\varphi \in \mathcal{L}^p$, but here we restrict our attention to $p=2$. A function φ belongs to $\mathcal{L}^2$ in the 1-periodic function

$$\varphi^\circ := \sum_{j \in \mathbb{Z}^s} |\varphi(\cdot - j)| \tag{1.1}$$

is in $\mathbf{L}^2([0..1)^s)$. The space $\mathcal{L}^2$ is a subspace of $\mathbf{L}^2(\mathbb{R}^s)$ that contains, for example, the class $\mathcal{E}^2$ of those functions decaying exponentially fast in

$\mathbf{L}^2(\mathbb{R}^s)$, i.e., the functions φ for which there is a q, $0<q<1$, and some constant so that

$$\|\varphi(\cdot-j)\|_{\mathbf{L}^2([0..1)^s)} \leqslant \text{const}\, q^{|j|} \qquad \text{for all} \quad j \in \mathbb{Z}^s,$$

and in particular, the compactly supported functions in $\mathbf{L}^2(\mathbb{R}^s)$. The function φ is *refinable* if it satisfies the refinement equation

$$\varphi(\cdot/2)/2^s = \varphi *' a_\varphi := \sum_{j \in \mathbb{Z}^s} \varphi(\cdot - j)\, a_\varphi(j) \tag{1.2}$$

for some sequence $a_\varphi \in l^1(\mathbb{Z}^s)$, while it is l^2-*stable* if

$$\|\varphi *' a\|_2 \geqslant \text{const}\|a\|_2 \qquad \forall a \in l^2(\mathbb{Z}^s). \tag{1.3}$$

The sequence a_φ is the *mask* of the refinement equation. From (1.2), we have

$$\hat{\varphi}(2\cdot) = H_\varphi\, \hat{\varphi}, \qquad \text{where } H_\varphi := \sum_{j \in \mathbb{Z}^s} a_\varphi(j) \exp(-ij\cdot). \tag{1.4}$$

Let $V_0 := \varphi *' l^2(\mathbb{Z}^s)$, and $V_\nu := \sigma_\nu V_0$, $\nu \in \mathbb{Z}$, where $\sigma_\nu f \mapsto f(2^\nu \cdot)$. If φ is in $\mathscr{L}^2$ and is refinable and l^2-stable, then Jia and Micchelli [JM$_2$] proved that the sequence of spaces $\{V_\nu\}_{\nu \in \mathbb{Z}}$ forms a *multiresolution approximation* to $\mathbf{L}^2(\mathbb{R}^s)$:

(i) $V_\nu \subset V_{\nu+1}$, $\nu \in \mathbb{Z}$.

(ii) $\bigcup_{\nu \in \mathbb{Z}} V_\nu$ is dense in $\mathbf{L}^2(\mathbb{R}^s)$ and $\bigcap_{\nu \in \mathbb{Z}} V_\nu = \{0\}$.

(iii) $f \in V_\nu \Leftrightarrow f(2\cdot) \in V_{\nu+1}$, $\forall \nu \in \mathbb{Z}$.

(iv) $f \in V_\nu \Rightarrow f(\cdot - 2^{-\nu} j) \in V_\nu$, $\forall \nu \in \mathbb{Z}$, and $\forall j \in \mathbb{Z}^s$.

(v) There is an unconditional basis for V_0.

A *pre-wavelet* is any function in the orthogonal complement of V_{-1} in V_0. A prewavelet has the property that

$$\psi(2^{\nu_1} \cdot - j_1) \perp \psi(2^{\nu_2} \cdot - j_2) \tag{1.5}$$

whenever $\nu_1 \neq \nu_2$ (orthogonality between levels) whereas it is a *wavelet* if it has this property whenever either $\nu_1 \neq \nu_2$ or $j_1 \neq j_2$.

2. Construction of Orthogonal Wavelets

In addition to the requirement that $\varphi \in \mathscr{L}^2$ be refinable, we assume that φ is skew-symmetric about some point $c_\varphi \in \mathbb{R}^s$:

$$\varphi(c_\varphi + x) = \overline{\varphi(c_\varphi - x)}, \qquad \forall x \in \mathbb{R}^s. \tag{2.1}$$

The symmetry and refinability interact to give further restrictions on the function φ: *The center of* φ, c_φ, *must be in* $\mathbb{Z}^s/2$. Indeed, this follows because the symmetry relation (2.1) on φ yields the following relation on H_φ:

$$\overline{H_\varphi(y)} = \exp(i2c_\varphi y)\, H_\varphi(y). \tag{2.2}$$

Thus, for any unit coordinate vector e_μ,

$$\begin{aligned}\overline{H_\varphi(y+2\pi e_\mu)} &= \exp(i2c_\varphi y)\, H_\varphi(y+2\pi e_\mu)\exp(2\pi i(2c_\varphi e_\mu))\\ &= \overline{H_\varphi(y)}\exp(2\pi i(2c_\varphi e_\mu)),\end{aligned}$$

which implies that $2c_\varphi e_\mu$ is an integer.

A wavelet decomposition from a multiresolution approximation of $\mathbf{L}^2(\mathbb{R}^s)$ is apparent from the following theorem. A simple method for the construction of wavelets from the refinement mask becomes clear in the proof and will be stated as (2.15) Corollary.

(2.3) THEOREM. *Let the refinable function* $\varphi \in \mathscr{L}^2$ *be skew-symmetric about the point* c_φ *and suppose its integer translates,* $\{\varphi(\cdot - j)\}_{j\in\mathbb{Z}^s}$, *form an orthonormal system in* $\mathbf{L}^2(\mathbb{R}^s)$, $s = 1, 2$, *or* 3. *Then the space* $V_0(\varphi) := \varphi *' l^2(\mathbb{Z}^s)$ *admits an orthogonal decomposition*

$$V_0(\varphi) = \bigoplus_{\mu\in\mathbb{Z}_2^s} O_\mu,$$

where $\mathbb{Z}_2^s = \{0, 1\}^s$. *Moreover, there exist* 2^s *functions* K_μ, $\mu\in\mathbb{Z}_2^s$, *such that for each* μ, $\{K_\mu(\cdot - j)\}_{j\in\mathbb{Z}^s}$ *is an orthogonal family,*

$$O_\mu = K_\mu(\cdot/2)/2^s *' l^2(2\mathbb{Z}^s),$$

and $K_0 := \varphi$.

Proof. The proof is the same as given in [RS] for box splines. We outline the proof here because the notation and ideas are needed for subsequent sections.

The first obvious requirement on the functions K_μ is that $K_\mu(\cdot/2)/2^s \in V_0$, or equivalently, that there are functions $H_\mu \in \mathbf{L}^2([0..2\pi]^s)$ such that

$$(K_\mu(\cdot/2)/2^s)^\wedge(y) = \hat{K}_\mu(2y) = H_\mu(y)\,\hat{\varphi}(y). \tag{2.4}$$

Using the fact that the orthonormality of the integer translates of φ is equivalent to

$$\sum_{j\in\mathbb{Z}^s} |\hat{\varphi}(\cdot + 2\pi j)|^2 = 1, \tag{2.5}$$

the orthogonality conditions

$$\delta(j)\,\delta(\mu_1-\mu_2)=\int_{\mathbb{R}^s} K_{\mu_1}(x-j)\,\overline{K_{\mu_2}(x)}\,dx \tag{2.6}$$

are equivalent to the relations

$$\sum_{\kappa\in\mathbb{Z}_2^s} H_{\mu_1}(\cdot+\pi\kappa)|^2=1, \qquad \mu_1\in\mathbb{Z}_2^s, \qquad \text{and}$$

$$\sum_{\kappa\in\mathbb{Z}_2^s} H_{\mu_1}(\,\cdot\,+\pi\kappa)\,\overline{H_{\mu_2}(\,\cdot\,+\pi\kappa)}=0, \qquad \mu_1\neq\mu_2. \tag{2.7}$$

The ability to find, for any $a\in l^2(\mathbb{Z}^s)$, sequences $b_k\in l^2(2\mathbb{Z}^s)$ such that

$$\varphi *' a=\sum_{\mu\in\mathbb{Z}_2^s} K_\mu(\cdot/2)/2^s *' b_\mu \tag{2.8}$$

is equivalent to the existence of π-periodic functions $B_\mu\in\mathbf{L}^2([0..\pi]^s)$ for any given $A\in\mathbf{L}^2([0..2\pi]^s)$ such that

$$A(y)=\sum_{\mu\in\mathbb{Z}_2^s} B_\mu(y)\,H_\mu(y),$$

which by π-periodicity gives rise to the system

$$A(y+\pi\kappa)=\sum_{\mu\in\mathbb{Z}_2^s} B_\mu(y)\,H_\mu(y+\pi\kappa), \qquad \kappa\in\mathbb{Z}_2^s. \tag{2.9}$$

The relations (2.7) also give the solvability of this system since they imply that the matrix

$$U=[H_\mu(\cdot+\pi\kappa)]_{\mu,\kappa\in\mathbf{Z}_2^s}$$

is a unitary matrix; i.e., $\bar{U}^TU=\mathbf{1}$.

Using (2.5) twice, first for the variable $2y$ and then for the variable $y+\pi\kappa$ with the relation $\hat{K}_0(2\cdot)=\hat{\varphi}(2\cdot)=H_\varphi\hat{\varphi}$ used in between, we find that H_φ satisfies the first of the relations (2.7). This means that the first condition in (2.7) is satisfied for any rotation and any shift of the functions H_φ or $\overline{H_\varphi}$ and leads to the choice of the functions H_μ as

$$H_\mu(y)=\exp(i\eta(\mu)\,y)\begin{cases} H_\varphi(y+\pi\mu), & \text{if} \quad 2c_\varphi\mu \text{ is even,}\\ \overline{H_\varphi(y+\pi\mu)}, & \text{if} \quad 2c_\varphi\mu \text{ is odd,}\end{cases}$$

where $\eta(\mu)\in\mathbb{Z}_2^s$. As was proved in [RS], the second relation in (2.7) is satisfied if the mapping $\eta\colon\mathbb{Z}_2^s\to\mathbb{Z}_2^s$ satisfies

$$\eta(0)=0, \qquad \text{and} \qquad (\eta(\mu_1)+\eta(\mu_2))(\mu_1+\mu_2) \text{ is odd when } \mu_1\neq\mu_2 \tag{2.11}$$

(where the first requirement is only to ensure $K_0 = \varphi$). For later use we state and prove this in a superficially more general form:

(2.12) LEMMA. *If η satisfies (2.11) and the functions*

$$G_\mu = \exp(i\eta(\mu)\cdot)\, G_\mu^*(y+\pi\mu) \qquad \textit{where} \quad G_\mu^*(y) = \begin{cases} G(y), & \textit{if} \quad 2c\mu \textit{ is even;} \\ \overline{G(y)}, & \textit{if} \quad 2c\mu \textit{ is odd,} \end{cases} \tag{2.13}$$

are defined for $\mu \in \mathbb{Z}_2^s$ from a 2π-periodic function G that satisfies

$$\overline{G(y)} = \exp(i2cy)\, G(y) \qquad \textit{with } c \in \mathbb{Z}^s/2,$$

then

$$\sum_{\kappa \in \mathbb{Z}_2^s} G_{\mu_1}(\cdot + \pi\kappa)\, \overline{G_{\mu_2}(\cdot + \pi\kappa)} = 0.$$

Proof of Lemma. We show that, under the assumptions, the terms in the last sum corresponding to κ and $\kappa + \mu_1 + \mu_2$ are equal except of opposite sign. The term corresponding to κ is

$$\begin{aligned} &G_{\mu_1}(\cdot + \pi\kappa)\, \overline{G_{\mu_2}(\cdot + \pi\kappa)} \\ &\quad = \exp(i(\eta(\mu_1) - \eta(\mu_2))\cdot)(-1)^{(\eta(\mu_1) - \eta(\mu_2))\kappa}\, G_{\mu_1}^*(\cdot + \pi(\mu_1 + \kappa)) \\ &\quad \times \overline{G_{\mu_2}^*(\cdot + \pi(\mu_2 + \kappa))}. \end{aligned}$$

On the other hand, we use the 2π-periodicity of G^* to write the term corresponding to $\kappa + \mu_1 + \mu_2$ as

$$\begin{aligned} &\exp(i(\eta(\mu_1) - \eta(\mu_2))\cdot)(-1)^{(\eta(\mu_2) - \eta(\mu_2))(\kappa + \mu_1 + \mu_2)} \\ &\quad \times G_{\mu_1}^*(\cdot + \pi(\mu_2 + \kappa))\, \overline{G_{\mu_2}^*(\cdot + \pi(\mu_1 + \kappa))}. \end{aligned}$$

These two terms will be equal except of opposite sign when η satisfies (2.11) if and only if

$$G_{\mu_1}^*(\cdot + \pi(\mu_1 + \kappa))\, \overline{G_{\mu_2}^*(\cdot + \pi(\mu_2 + \kappa))} = G_{\mu_1}^*(\cdot + \pi(\mu_2 + \kappa))\, \overline{G_{\mu_2}^*(\cdot + \pi(\mu_1 + \kappa))}.$$

If one of $2c\mu_1$ and $2c\mu_2$ is even and the other odd, then the last relation follows directly from the definition of G_μ^* since $G_{\mu_1}^*$ and $\overline{G_{\mu_2}^*}$ are either both G or both $\bar{G}$.

When one of $G_{\mu_1}^*$ and $\overline{G_{\mu_2}^*}$ is G and the other is $\bar{G}$, the proof requires the assumed relation between G and $\bar{G}$. Then, if both $2c\mu_1$ and $2c\mu_2$ are even, we have

$$\begin{aligned} &G^*_{\mu_1}(\cdot+\pi(\mu_1+\kappa))\,\overline{G^*_{\mu_2}(\cdot+\pi(\mu_2+\kappa))}\\ &\quad=\exp(i2c\cdot)(-1)^{2c(\kappa+\mu_2)}\,G(\cdot+\pi(\kappa+\mu_1))\,G(\cdot+\pi(\kappa+\mu_2))\\ &\quad=\exp(i2c\cdot)(-1)^{2c(\kappa+\mu_1)}\,G(\cdot+\pi(\kappa+\mu_2))\,G(\cdot+\pi(\kappa+\mu_1))\\ &\quad=G^*_{\mu_1}(\cdot+\pi(\mu_2+\kappa))\,\overline{G^*_{\mu_2}(\cdot+\pi(\mu_1+\kappa))}. \end{aligned}$$

The case when both $2c\mu_1$ and $2c\mu_2$ are odd is similar. ∎

Returning to the proof of the theorem, we recall that the mappings η can be given explicitly for dimensions $s=1,2,3$. For $s=1$, $\eta(0)=0$ and $\eta(1)=1$. For $s=2$, the mapping can be given by

$$\begin{matrix} (0,0)\mapsto(0,0) & (0,1)\mapsto(0,1)\\ (1,0)\mapsto(1,1) & (1,1)\mapsto(1,0). \end{matrix} \tag{2.14}$$

Finally, for $s=3$, the mapping

$$\begin{matrix} (0,0,0)\mapsto(0,0,0), & (1,0,0)\mapsto(1,1,0), & (0,1,0)\mapsto(0,1,1),\\ (1,1,0)\mapsto(1,0,0), & (0,0,1)\mapsto(1,0,1), & (1,0,1)\mapsto(0,0,1),\\ (0,1,1)\mapsto(0,1,0), & (1,1,1)\mapsto(1,1,1) & \end{matrix}$$

satisfies the property (2.11). However, as remarked in [RS], it is impossible to find such a mapping for dimensions $s>3$.

Finally, each H_μ has absolutely summable Fourier series (since H_φ does) and so $K_\mu(\cdot/2)*'\,l^2(2\mathbb{Z}^s)\subseteq V_0$. The proof is complete. ∎

The form (2.10) of the functions H_μ makes it particularly easy to determine the functions K_μ from φ and its refinement mask.

(2.15) COROLLARY. *If a_φ is the refinement mask in the equation* (1.2), *then the functions K_μ are given by*

$$K_\mu(\cdot/2)/2^s=\varphi *' a_\mu,\qquad \mu\in\mathbb{Z}_2^s,$$

where for $\mu\neq 0$

$$a_\mu(j)=(-1)^{j\mu+1}\,a^*((-1)^{2c_\varphi\mu}\,(j+\eta(\mu))),$$

and

$$a^*=\begin{cases} a_\varphi, & \text{if } 2c_\varphi\mu \text{ is even};\\ \overline{a_\varphi}, & \text{if } 2c_\varphi\mu \text{ is odd}. \end{cases}$$

The function K_μ, $\mu\in\mathbb{Z}_2^s$, is skew-symmetric or anti skew-symmetric about the point

$$c_\mu:=(c_\varphi(1+(-1)^{2c_\varphi\mu})-\eta(\mu))/2;\qquad i.e.,\ K_\mu(c_\mu+x)=(-1)^{2c_\varphi\mu}\overline{K_\mu(c_\mu-x)}.$$

If $\varphi \in \mathscr{E}^2$, then the coefficient sequences have exponential decay; i.e.,

$$|a_\mu(j)| \leqslant \text{const} \exp(-|j|/\text{const}), \qquad \forall j \in \mathbb{Z}^s \text{ and } \mu \in \mathbb{Z}_2^s$$

for some positive constants and each of the functions $K_\mu \in \mathscr{E}^2$.

Proof. The first statement is an immediate consequence of Eq. (2.10) and the fact that a_φ is the sequence of Fourier coefficients for H_φ, while the last statement follows from observations in [JM$_2$].

By the skew-symmetry of φ, or by (2.2), we have $\overline{a_\varphi(-j)} = a_\varphi(j + 2c_\varphi)$. Hence, from the definition of a_μ

$$\overline{a_\mu(-j)} = (-1)^{2c_\varphi\mu} a_\mu(j + 2(-1)^{2c_\varphi\mu} c_\varphi - 2\eta(\mu)).$$

Using this last relation and the skew-symmetry of φ, we find

$$\begin{aligned}
K_\mu(c_\mu + x) &= 2^s \sum_{j \in \mathbb{Z}^s} a_\mu(j)\, \varphi(2c_\mu + 2x - j) \\
&= 2^s \sum_{j \in \mathbb{Z}^s} a_\mu(j)\, \overline{\varphi(2c_\varphi - 2c_\mu - 2x + j)} \\
&= 2^s \overline{\sum_{j \in \mathbb{Z}^s} \overline{a_\mu(-j)}\, \varphi(2c_\varphi - 2c_\mu - 2x - j)} \\
&= 2^s (-1)^{2c_\varphi\mu} \overline{\sum_{j \in \mathbb{Z}^s} a_\mu(j + 2(-1)^{2c_\varphi\mu} c_\varphi - 2\eta(\mu))\, \varphi(2c_\varphi - 2c_\mu - 2x - j)} \\
&= (-1)^{2c_\varphi\mu} \overline{K_\mu(c_\mu - x)}. \quad \blacksquare
\end{aligned}$$

When the integer translates of $\varphi \in \mathscr{L}^2$ are not orthogonal but φ is nonetheless l^2-stable as well as refinable, the usual way of obtaining K_0 from φ is the following: The l^2-stability implies that the symbol

$$P := \sum_{j \in \mathbb{Z}^s} |\hat{\varphi}(\cdot + 2\pi j)|^2 = \sum_{j \in \mathbb{Z}^s} \varphi * \overline{\varphi(-\cdot)}(j) \exp(-ij\cdot) \tag{2.16}$$

does not vanish. We note that for skew-symmetric φ,

$$P := \sum_{j \in \mathbb{Z}^s} \varphi^c * \overline{\varphi^c}(j) \exp(-ij\cdot), \tag{2.17}$$

where $\varphi^c = \varphi(c_\varphi + \cdot)$. The function K_0 is then given via its Fourier transform

$$\hat{K}_0 := \frac{\hat{\varphi}}{\sqrt{P}}. \tag{2.18}$$

Then $K_0 = \varphi *' a_K$ with $a_K \in l^1(\mathbb{Z}^s)$ and K_0 is refinable since

$$\hat{K}_0(2y) = \frac{\hat{\varphi}(2y)}{\sqrt{P(2y)}} = \frac{H_\varphi(y)\sqrt{P(y)}}{\sqrt{P(2y)}}\, \hat{K}_0(y). \tag{2.19}$$

Moreover, $V_0 = K_0 *' l^2(\mathbb{Z}^s)$ (since $\varphi = K_0 *' (\sqrt{P})^\vee$). Therefore, K_0 generates the multiresolution approximation $\{V_\nu\}$. Since K_0 is skew-symmetric about c_φ as well, (2.3) Theorem and (2.15) Corollary hold with φ replaced by K_0 and a_φ replaced by the refinement mask for K_0 as determined by (2.19).

3. Construction of Compactly Supported Pre-Wavelets

In this section we assume that $\varphi \in \mathbf{L}^2(\mathbb{R}^s)$, $s = 1, 2, 3$, is compactly supported, l^2-stable, skew-symmetric about a point c_φ, and satisfies the refinement equation (1.2) with a *finitely supported mask*. The fact that then the 2π-periodic functions P and H_φ defined in (2.16) and (1.4) are trigonometric polynomials plays an important role in our construction.

The pre-wavelets φ_μ, $\mu \in \mathbb{Z}_2^s \backslash 0$, are defined via their Fourier transforms,

$$\hat{\varphi}_\mu(2y) := H_\mu(y)\, \hat{\varphi}(y), \qquad \mu \in \mathbb{Z}_2^s \backslash 0, \tag{3.1}$$

where

$$H_\mu(y) = \exp(i\eta(\mu)\, y)\, P(y + \pi\mu) \begin{cases} H_\varphi(y + \pi\mu), & \text{if } 2c_\varphi\mu \text{ is even;} \\ \overline{H_\varphi(y + \pi\mu)}, & \text{if } 2c_\varphi\mu \text{ is odd.} \end{cases} \tag{3.2}$$

It is interesting to note that (3.2) is exactly analogous to (2.10) since $P = 1$ when the integer translates are orthonormal. If we also define $H_0 := H_\varphi$ and $\varphi_0 := \varphi$, then (3.1) is also true for $\mu = 0$.

(3.3) Theorem. *Under the above assumptions on φ, the functions $\varphi_\mu(\cdot/2)/2^s$, $\mu \in \mathbb{Z}_2^s \backslash 0$, are compactly supported pre-wavelets.*

Proof. It follows from (3.1) that each $\varphi_\mu(\cdot/2)/2^s$ is in V_0 and, since the functions H_μ are trigonometric polynomials, the functions φ_μ are compactly supported.

For the orthogonality between levels, we find

$$\begin{aligned} &\langle \varphi_0(\cdot/2)/2^s, \varphi_\mu(\cdot/2 - j)/2^s \rangle \\ &\quad = \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \hat{\varphi}(2y)\, \overline{\hat{\varphi}(2y)}\, \exp(2ijy)\, dy \\ &\quad = \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} |\hat{\varphi}(y)|^2\, H_\varphi(y)\, \overline{H_\mu(y)}\, \exp(2ijy)\, dy \end{aligned}$$

$$= \frac{1}{(2\pi)^s} \int_{[0..2\pi)^s} P(y)\, H_\varphi(y)\, \overline{H_\mu(y)} \exp(2ijy)\, dy$$

$$= \frac{1}{(2\pi)^s} \int_{[0..\pi)^s} \left(\sum_{\kappa \in \mathbb{Z}_2^s} P(y+\pi\kappa)\, H_0(y+\pi\kappa)\, \overline{H_\mu(y+\pi\kappa)} \right) \exp(2ijy)\, dy$$

$$= 0,$$

by (2.12) Lemma with $G = PH_0 = PH_\varphi$. ∎

The last proof shows the motivation for our choice of H_μ. In order to retain the simple form as a rotation and translation of one function and to apply (2.12) Lemma, we must choose some function G. But the orthogonality between levels automatically brings in $P(y)\, H_\varphi(y)$ when the integration is reduced from $\mathbb{R}^s$ to $[0..2\pi]^s$ by summation and this makes $G = P(y)\, H_\varphi(y)$ the logical choice.

Next we show that the pre-wavelets $\varphi_\mu(\cdot/2)/2^s$, $\mu \in \mathbb{Z}_2^s \backslash 0$, together with $\varphi(\cdot/2)/2^s$ generate an unconditional base for V_0 in the following sense: Given $a \in l^2(\mathbb{Z}^s)$, there are sequences $b_\mu \in l^2(2\mathbb{Z}^s)$ such that

$$\varphi *' a = \sum_{\mu \in \mathbb{Z}_2^s} \varphi_\mu(\cdot/2)/2^s *' b_\mu \tag{3.4}$$

and there are positive constants C_1 and C_2 so that

$$C_1 \sum_{\mu \in \mathbb{Z}_2^s} \|b_\mu\|_2 \leqslant \|\varphi *' a\|_2 \leqslant C_2 \sum_{\mu \in \mathbb{Z}_2^s} \|b_\mu\|_2. \tag{3.5}$$

Towards this end, we first prove

(3.6) Proposition. *Let A be an arbitrary subset of $\mathbb{Z}_2^s$. If φ is l^2-stable, then the set of functions $\{\varphi_\mu(\cdot/2)\}_{\mu \in A}$, defined by* (3.1), *generate an unconditional base for the space*

$$W_A := \left\{ \sum_{\mu \in A} \varphi_\mu(\cdot/2) *' a_\mu : a_\mu \in l^2(2\mathbb{Z}^s) \right\}$$

if and only if the set of vectors

$$(H_\mu(y+\pi\kappa))_{\kappa \in \mathbb{Z}_2^s}, \qquad \mu \in A$$

are linearly independent for any $y \in \mathbb{R}^s$.

Proof. For the functions to be an unconditional base requires

$$C_1 \sum_{\mu \in \mathbb{Z}_2^s} \|a_\mu\|_2 \leqslant \left\| \sum_{\mu \in \mathbb{Z}_2^s} \varphi_\mu(\cdot/2)/2^s *' a_\mu \right\|_2 \leqslant C_2 \sum_{\mu \in \mathbb{Z}_2^s} \|a_\mu\|_2.$$

The upper bound is always true for compactly supported $\mathbf{L}^2(\mathbb{R}^s)$ functions. The lower estimate is the stability of several functions as studied by Jia and Micchelli [JM_1, JM_2]). By [JM_1, Theorem 4], the stability is equivalent to showing that for any $y \in \mathbb{R}^s$, the sequences $\{\hat{\varphi}_\mu(y+2\pi j)\}_{j \in \mathbb{Z}^s}$, $\mu \in A$ are linearly independent. But $\{\hat{\varphi}_\mu(y+2\pi j)\}_{j \in \mathbb{Z}^s}$, $\mu \in A$ are linearly dependent for some $y \in \mathbb{R}^s$ if and only if there are complex numbers m_μ, $\mu \in A$, not all zero such that

$$\sum_{\mu \in A} m_\mu \hat{\varphi}_\mu(y+2\pi j) = 0 \qquad \forall j \in \mathbb{Z}^s.$$

From (3.1), this is equivalent to

$$\sum_{\mu \in A} m_\mu H_\mu(y/2+\pi j)\, \hat{\varphi}(y/2+\pi j) = 0 \qquad \forall j \in \mathbb{Z}^s,$$

or

$$\hat{\varphi}(y/2+\pi\kappa+2\pi j^*) \sum_{\mu \in A} m_\mu H_\mu(y/2+\pi\kappa) = 0 \qquad \forall j^* \in \mathbb{Z}^s,\ \kappa \in \mathbb{Z}_2^s.$$

Since $\{\varphi(\cdot/j)\}_{j \in \mathbb{Z}^s}$ is an unconditional basis of V_0, for each $\kappa \in \mathbb{Z}_2^s$, there is an $j^* = j^*(\kappa)$ such that $\hat{\varphi}(y/2+\pi\kappa+2\pi j^*) \neq 0$. Thus,

$$\sum_{\mu \in A} m_\mu H_\mu(y/2+\pi\kappa) = 0, \qquad \forall \kappa \in \mathbb{Z}_2^s. \quad \blacksquare$$

(3.7) COROLLARY. *The set of functions* $\{\varphi_\mu(\cdot/2)\}$, $\mu \in \mathbb{Z}_2^s$, *generates an unconditional basis for* $W_{\mathbb{Z}_2^s}$. *In particular,*

$$W_{\mathbb{Z}_2^s} = V_{-1} \oplus W_{\mathbb{Z}_2^s \setminus 0}$$

is an orthogonal decomposition and $\{\varphi_\mu(\cdot/2)\}$, $\mu \in \mathbb{Z}_2^s \setminus 0$, *generates an unconditional basis for* $W_{\mathbb{Z}_2^s \setminus 0}$.

Proof. In the proof of (3.3), we have already established that $V_{-1} \perp W_{\mathbb{Z}_2^s \setminus 0}$. Hence, it only remains to show that $\{\varphi_\mu(\cdot/2)\}$, $\mu \in \mathbb{Z}_2^s \setminus 0$, generates an unconditional basis for $W_{\mathbb{Z}_2^s \setminus 0}$. By (3.6) Proposition, it suffices to show that the matrix

$$U_1 := [H_\mu(y+\pi\kappa)]_{\kappa \in \mathbb{Z}_2^s,\, \mu \in \mathbb{Z}_2^s \setminus 0}$$

has full rank. We observe that by (2.12) Lemma with $G = PH_\varphi$, $\overline{U_1}^T U_1$ is a diagonal matrix with diagonal entries

$$\sum_{\kappa \in \mathbb{Z}_2^s} P(y+\pi\mu+\pi\kappa)^2\, |H_\varphi(y+\pi\mu+\pi\kappa)|^2.$$

Since P^2 is strictly positive, the proof will be complete once we have established

(3.8) LEMMA. *Under the above assumptions on φ, the π-periodic function*

$$\sum_{\mu \in \mathbb{Z}_2^s} |H_\varphi(y+\mu\pi)|^2$$

is strictly positive.

Proof. This follows from

$$\begin{aligned} 0 < P(2y) &= \sum_{j \in \mathbb{Z}^s} |\hat{\varphi}(2y+2\pi j)|^2 \\ &= \sum_{j \in \mathbb{Z}^s} |H_\varphi(y+\pi j)|^2 \, |\hat{\varphi}(y+\pi j)|^2 \\ &= \sum_{\mu \in \mathbb{Z}_2^s} |H_\varphi(y+\pi\mu)|^2 \, P(y+\mu\pi). \quad \blacksquare \end{aligned}$$

(3.9) THEOREM. $V_0 = W_{\mathbb{Z}_2^s} = V_{-1} \oplus W_{\mathbb{Z}_2^s \backslash 0}$ *is an orthogonal decomposition.*

Proof. It suffices to prove that for an arbitrary $\varphi *' a \in V_0$, $a \in l(\mathbb{Z}^s)$, there are $b_\kappa \in l^2(2\mathbb{Z}^s)$ such that

$$\varphi *' a = \sum_{\kappa \in \mathbb{Z}_2^s} \varphi_\kappa(\cdot/2)/2^s *' b_\kappa.$$

As in the proof of (2.3) Theorem, this is equivalent to the solvability for $B_\kappa \in \mathrm{L}^2([0..\pi]^s)$ in the system of equations

$$A(\cdot + \kappa\pi) = \sum_{\mu \in \mathbb{Z}_2^s} B_\mu H_\mu(\cdot + \kappa\pi), \qquad \kappa \in \mathbb{Z}_2^s,$$

with coefficient matrix $U = [H_\mu(\cdot + \pi\kappa)]_{\mu,\kappa \in \mathbb{Z}_2^s}$. But (3.7) Corollary and (3.6) Proposition imply that U has full rank. $\blacksquare$

Again we have the nice property that all the coefficients can be derived from a given sequence:

(3.10) COROLLARY. *Let*

$$\beta_0 = (\varphi * \overline{\varphi(-\cdot)})_| * a_\varphi = (\varphi^c * \varphi^c)_| * a_\varphi,$$

where the notation $f_|$ denotes the restriction of the function f to $\mathbb{Z}^s$. Then

$$\varphi_\mu(\cdot/2)/2^s = \varphi *' \beta_\mu, \qquad \mu \neq 0,$$

with

$$\beta_\mu(j) = (-1)^{j\mu+1} \beta^*((-1)^{2c_\varphi \mu}(j+\eta(\mu))),$$

where

$$\beta^* = \begin{cases} \beta_0, & \text{if } 2c_\varphi\mu \text{ is even;} \\ \overline{\beta_0}, & \text{if } 2c_\varphi\mu \text{ is odd.} \end{cases}$$

Moreover, the function φ_μ *is skew-symmetric or skew anti-symmetric about the point*

$$c_\mu = (c_\varphi(1 + (-1)^{2c_\varphi\mu}) - \eta(\mu))/2; \qquad \text{i.e.,} \quad \varphi_\mu(c_\mu + x) = (-1)^{2c_\varphi\mu}\overline{\varphi_\mu(c_\mu - x)}, \quad \mu \in \mathbb{Z}_2^s.$$

4. Orthogonalization within the Same Level

In this section we regain some of the orthogonality that was lost by requiring a decomposition by compactly supported functions. We present two alternatives. The first is the usual type of Gram–Schmidt procedure to gain orthogonality between the generating functions φ_μ while preserving the compactness of support. The second approach will be to give an equivalent inner product defined in terms of the symbol P and the refinement mask symbol H_φ under which the integer translates of the pre-wavelets become an orthonormal system for $W_{\mathbb{Z}_2^s\backslash 0}$.

For the orthogonalization of the functions φ_μ, $\mu \in \mathbb{Z}_2^s\backslash 0$, Jia and Micchelli [JM$_2$, Sect. 4] have given a procedure which we interpret here in our notation. For this, we select some definite order and label the functions by integers 1 to $2^s - 1$ for this process. We take $\psi_1 = \varphi_1$ and proceed inductively as follows. If $\psi_1, ..., \psi_n$ have been constructed, then we set

$$\begin{aligned} c_n &= c_{n-1} * (\psi_n * \overline{\psi_n(-\cdot)})_| \\ c_{n,m} &= c_{n-1,m} * (\psi_n * \overline{\psi_n(-\cdot)})_|, \qquad 1 \leqslant m < n, \\ c_{n,n} &= c_{n-1}, \end{aligned}$$

and define

$$\psi_{n+1} = \varphi_{n+1} *' c_n - \sum_{m=1}^{n} \psi_m *' (c_{n,m} * (\varphi_{n+1} * \overline{\psi_m(-\cdot)})_|).$$

The orthogonal pre-wavelets so constructed obviously have compact support, but this support increases with each stage of the procedure. The orthogonalization process also requires that we know the values on the integer lattice of a (continuous) convolution of any two of the pre-wavelets. Finally, the nice property that all sequences can be found easily from one sequence is lost (see (2.15) Corollary and (3.10) Corollary).

In the second approach, we construct an inner product under which the collection of functions

$$\{\varphi_\mu(2^{-1} \cdot - j)/2^{s/2}\}_{\mu \in \mathbb{Z}_2^s, j \in \mathbb{Z}^s}$$

is an orthonormal family. For this purpose, we define

$$\langle f, g\rangle_{-1} = \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \frac{\hat{f}(y)\,\overline{\hat{g}(y)}}{Q(y)}\, dy, \qquad f, g \in W_{\mathbb{Z}_2^s\setminus 0}, \tag{4.1}$$

where

$$Q(y) = P(y) \sum_{\kappa \in \mathbb{Z}_2^s} P^2(y+\pi\kappa)\, |H_\varphi(y+\pi\kappa)|^2. \tag{4.2}$$

To carry this idea to other dyadic levels and for convenience of notation, we set $W_{-1} := W_{\mathbb{Z}_2^s\setminus 0}$ and define

$$W_\nu := \{f : f(2^{-\nu-1}\cdot) \in W_{\mathbb{Z}_2^s\setminus 0} = W_{-1}\}. \tag{4.3}$$

Then

$$V_{\nu+1} = V_\nu \oplus W_\nu, \qquad \forall \nu \in \mathbb{Z} \Rightarrow V_N = \bigoplus_{\nu \leqslant N-1} W_\nu$$

is an orthogonal decomposition. On the space W_ν, we define the inner product

$$\begin{aligned}\langle f, g\rangle_\nu &= \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \frac{\hat{f}(y)\,\overline{\hat{g}(y)}}{Q(2^{-\nu-1}y)}\, dy, \\ &= 2^{-(\nu+1)s} \langle f(2^{-\nu-1}\cdot), g(2^{-\nu-1}\cdot)\rangle_{-1} \qquad f, g \in W_\nu.\end{aligned}$$

(4.5) THEOREM. *The functions* $\{2^{\nu s/2}\varphi_\mu(2^\nu\cdot - j)\}$, $\mu \in \mathbb{Z}_2^s\setminus 0$, $j \in \mathbb{Z}^s$ *are an orthonormal basis for the space* W_ν *with the inner product* $\langle\ ,\ \rangle_\nu$.

Proof. For $\nu = -1$, we have

$$\begin{aligned}&\frac{1}{2^s}\langle \varphi_{\mu_1}(2^{-1}\cdot), \varphi_{\mu_2}(2^{-1}\cdot - j)\rangle_{-1} \\ &= \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \frac{2^s\,\hat{\varphi}_{\mu_1}(2y)\,\overline{\hat{\varphi}_{\mu_2}(2y)}\exp(i2jy)}{Q(y)}\, dy \\ &= \frac{1}{\pi^s} \int_{\mathbb{R}^s} \frac{H_{\mu_1}(y)\,\overline{H_{\mu_2}(y)}\,|\hat{\varphi}(y)|^2\exp(i2jy)}{Q(y)}\, dy \\ &= \frac{1}{\pi^s} \int_{[0..2\pi]^s} \frac{H_{\mu_1}(y)\,\overline{H_{\mu_2}(y)}\,P(y)\exp(i2jy)}{P(y)\sum_{\kappa\in\mathbb{Z}_2^s} P^2(y+\pi\kappa)\,|H_\varphi(y+\pi\kappa)|^2}\, dy \\ &= \frac{1}{\pi^s} \int_{[0..\pi]^s} \frac{\sum_{\kappa\in\mathbb{Z}_2^s} H_{\mu_1}(y+\pi\kappa)\,\overline{H_{\mu_2}(y+\pi\kappa)}\exp(i2jy)}{\sum_{\kappa\in\mathbb{Z}_2^s} P^2(y+\pi\kappa)\,|H_\varphi(y+\pi\kappa)|^2}\, dy \\ &= \begin{cases} 1, & \text{if } \mu_1 = \mu_2 \text{ and } j = 0 \\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

by (2.12) Lemma. The proof for the other values of ν follows by change of variable. ∎

The inner products on each W_ν can be used to define an inner product on the whole of $\bigoplus_{\nu \in \mathbb{Z}} W_\nu$. Indeed, we may define

$$\langle f, g \rangle_\infty := \sum_{\nu \in \mathbb{Z}} \langle f_\nu, g_\nu \rangle_\nu, \tag{4.6}$$

where $f, g \in \mathbf{L}^2(\mathbb{R}^s)$ and

$$f = \sum_{\nu \in \mathbb{Z}} f_\nu, \qquad g = \sum_{\nu \in \mathbb{Z}} g_\nu, \qquad f_\nu, g_\nu \in W_\nu.$$

(4.7) PROPOSITION. *The inner product $\langle\, , \rangle_\infty$ has the following properties*:

(i) *The functions $\{2^{\nu s/2} \varphi_\mu(2^\nu \cdot - j)\}$, $\mu \in \mathbb{Z}_2^s$, $j \in \mathbb{Z}^s$, $\nu \in \mathbb{Z}$, form an orthonormal system under $\langle\, , \rangle_\infty$.*

(ii) *If $f = \sum_{\nu \in \mathbb{Z}} f_\nu$ and $g \in \sum_{\nu \in \mathbb{Z}} g_\nu$ are functions in $\mathbf{L}^2(\mathbb{R}^s)$, and*

$$f_\nu = 2^{\nu s/2} \sum_{\mu \in \mathbb{Z}_2^s \backslash 0} \varphi_\mu(2^\nu \cdot) *' a_{f, \nu, \mu}, \qquad a_{f, \nu, \mu} \in l^2(2^{-\nu} \mathbb{Z}^s)$$

$$g_\nu = 2^{\nu s/2} \sum_{\mu \in \mathbb{Z}_2^s \backslash 0} \varphi_\mu(2^\nu \cdot) *' a_{g, \nu, \mu}, \qquad a_{g, \nu, \mu} \in l^2(2^{-\nu} \mathbb{Z}^s)$$

then

$$\langle f, g \rangle_\infty = \sum_{\nu \in \mathbb{Z}} \sum_{\mu \in \mathbb{Z}_2^s \backslash 0} \sum_{j \in 2^{-\nu} \mathbb{Z}^s} a_{f, \nu, \mu}(j)\, \overline{a_{g, \nu, \mu}(j)}.$$

(iii) *For any ν and $f \in V_\nu$, there are absolute positive constants such that*

$$C_1 \langle f, f \rangle_\infty \leqslant \|f\|_{\mathbf{L}^2(\mathbb{R}^s)} \leqslant C_2 \langle f, f \}_\infty.$$

Proof. The first two statements are clear. For (iii), let N be a fixed integer and $f \in V_N$. Then $f = \sum_{\nu \leqslant N-1} f_\nu$ where $f_\nu \in W_\nu$ and

$$\|f\|_{\mathbf{L}^2(\mathbb{R}^s)} = \sum_{\nu \leqslant N-1} \|f_\nu\|_{\mathbf{L}^s(\mathbb{R}^s)} \qquad \text{while} \quad \langle f, f \rangle_\infty = \sum_{\nu \leqslant N-1} \langle f_\nu, f_\nu \rangle_\nu.$$

Hence, we need to compare $\|f_\nu\|_{\mathbf{L}^2(\mathbb{R}^s)}$ and $\langle f_\nu, f_\nu \rangle_\nu$. But $f_\nu \in W_\nu$ implies

$$f_\nu = \sum_{\mu \in \mathbb{Z}_2^s \backslash 0} 2^{\nu s/2} \varphi_\mu(2^\nu \cdot) *' a_{\nu, \mu}, \qquad a_{\nu, \mu} \in l^2(2^{-\nu} \mathbb{Z}^s).$$

Thus,

$$\langle f_\nu, f_\nu \rangle_\nu = \sum_{\mu \in \mathbb{Z}_2^s \backslash 0} \|a_{\nu, \mu}\|_{l^2(2^{-\nu} \mathbb{Z}^s)}.$$

On the other hand,

$$2^{-(\nu+1)s/2} f_\nu(2^{-\nu-1}\cdot) = \sum_{\mu \in \mathbb{Z}_2^s \backslash 0} \varphi_\mu(\cdot/2)/2^s *' a_\mu^* =: h \in W_{\mathbb{Z}_2^s \backslash 0},$$

where $a_\mu^* = 2^{s/2} a_{\nu,\mu}(2^{\nu+1}\cdot) \in l^2(2\mathbb{Z}^s)$ with $\|a_\mu^*\|_{l^2(2\mathbb{Z}^s)} = 2^{s/2} \|a_{\nu,\mu}\|_{l^2(2^{-\nu}\mathbb{Z}^s)}$. Since $\|f_\nu\|_{\mathbf{L}^2(\mathbb{R}^s)} = \|h\|_{\mathbf{L}^2(\mathbb{R}^s)}$ and the functions $\varphi_\mu(\cdot/2)/2^s$ generate an unconditional basis for $W_{\mathbb{Z}_2^s \backslash 0}$, there are absolute positive constants such that

$$C_1 \|a_{\nu,\mu}\|_{l^2(2^{-\nu}\mathbb{Z}^s)} \leqslant \|f_\nu\|_{\mathbf{L}^2(\mathbb{R}^s)} \leqslant C_2 \|a_{\nu,\mu}\|_{l^2(2^{-\nu}\mathbb{Z}^s)}$$

and the result follows. ▌

Therefore,

(4.8) THEOREM. *The inner product* $\langle\,,\rangle_\infty$ *provides an equivalent norm on* $\mathbf{L}^2(\mathbb{R}^s)$ *under which the functions* $\{2^{\nu s/2}\varphi_\mu(2^\nu\cdot - j)\}$, $\mu \in \mathbb{Z}_2^s \backslash 0$, $j \in \mathbb{Z}^s$, $\nu \in \mathbb{Z}$, *form an orthonormal basis.*

In this second approach, the difficulty was hidden in the inner product. Although the inner product is defined in terms of the Fourier transforms, it can be computed in terms of f. To find the coefficients in the expansion for a given f at level -1 it is possible to proceed as follows: We find the Fourier coefficients of $1/Q$. Since Q is a strictly positive trigonometric polynomial, the coefficient sequence a_Q has exponential decay. Define the functions

$$R_\mu(x) = \sum_{j \in \mathbb{R}^s} \overline{\varphi_\mu((x-j)/2)/2^{s/2}}\, a_Q(j), \qquad \mu \in \mathbb{Z}_2^s \backslash 0.$$

Then

$$\langle f, \varphi_\mu(\cdot/2 - j)/2^{s/2} \rangle_{-1} = \int_{\mathbb{R}^s} f(x)\, R_\mu(x+2j)\, dx.$$

From (4.4), we see that all coefficients of f can be calculated from the convolution of dilates of f with translates of one set of functions.

5. BOX SPLINE EXAMPLES

The main examples of functions for which our analysis holds come from box splines. Box splines are defined on $\mathbb{R}^s$ for a given set of directions as specified by an $s \times n$ matrix Ξ via the distributional relation

$$\langle M_\Xi, f \rangle := \int_{[0..1)^n} f(\Xi t)\, dt, \qquad f \in C(\mathbb{R}^s),$$

or via its Fourier transform

$$\hat{M}_{\Xi} = \prod_{\xi \in \Xi} \frac{1 - \exp(-i\xi y)}{i\xi y} \tag{5.1}$$

(where $\xi \in \Xi$ means that the product is over all columns of Ξ). For $s = 2$, the primary examples of l^2-stable box splines are those for which the columns of the matrix Ξ consist of

$$\zeta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \zeta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \zeta_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

repeated n_1, n_2, and n_3 times respectively (at least two different directions must appear). These box splines will be denoted by M_{n_1, n_2, n_3} for brevity. We refer the reader to [RS] and [JM_2] and the references therein for additional information about the relationship of box splines to wavelets. As was mentioned in the introduction, Jia and Micchelli [JM_2] already gave pre-wavelets for the box spline $M_{1,1,1}$ and indicated a construction for $M_{2,2,1}$. Also, Chui, Stöckler, and Ward [CSW] have given a detailed account of the $M_{1,1,1}$ pre-wavelet that would result from our construction and even provided the Gram–Schmidt type orthogonalization. We give two examples by providing the important elements for deriving the wavelets; the values of $(\varphi^c * \varphi^c)_|$, the refinement mask a_φ, and the "mother" mask β_0 from which the pre-wavelets can be obtained by translation and change of sign pattern via (3.10) Corollary. It follows from (5.1) that for box splines, $M_{\Xi}^c * M_{\Xi}^c = M_{\Xi \cup \Xi}^c$ (where $\Xi \cup \Xi$ is the $s \times 2n$ matrix of columns from both matrices), and that the refinement mask can be found easily as the coefficients of

$$H_{M_\Xi}(y) = 2^{-n} \prod_{\xi \in \Xi} (1 + \exp(-i\xi y)).$$

Thus, the simplicity of the sequence formulation of the construction becomes evident.

The Box Spline $M_{2,2,1}$. The box spline $M_{2,2,1}$ is a C^1 piecewise cubic function that has support contained in $[0..3]^2$ and satisfies the stability condition. This box spline is symmetric about the point $c = (\frac{3}{2}, \frac{3}{2})$ and $2c\mu$ is odd for $\mu = (0, 1)$ and $(1, 0)$. The latter must be taken into account when finding β_μ from β_0.

The refinement mask, a_φ, for $\varphi = M_{2,2,1}$ has support contained in $[0..3]^2 \cap \mathbb{Z}^2$ with values on that square given by the matrix

$$a_\varphi = \frac{1}{32} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 4 & 5 & 2 \\ 2 & 5 & 4 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

The sequence $(\varphi^c * \varphi^c)_|$ for $\varphi = M_{2,2,1}$ has support contained in $[-3..3]^2 \cap \mathbb{Z}^2$ with values

$$(M^c_{2,2,1} * M^c_{2,2,1})_| = \frac{1}{10080} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 31 & 47 & 5 & 0 \\ 0 & 1 & 178 & 1144 & 814 & 47 & 0 \\ 0 & 31 & 1144 & 3194 & 1144 & 31 & 0 \\ 0 & 47 & 814 & 1144 & 178 & 1 & 0 \\ 0 & 5 & 47 & 31 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Convolving these two sequences gives the mask β_0 with support contained in the square $[-2..5]^2 \cap \mathbb{Z}^2$:

$$\beta_0 = \frac{1}{322560} \times \begin{bmatrix} 0 & 0 & 1 & 33 & 110 & 130 & 57 & 5 \\ 0 & 2 & 215 & 1677 & 3630 & 3136 & 1027 & 57 \\ 1 & 215 & 3134 & 12048 & 18425 & 12303 & 3136 & 130 \\ 33 & 1677 & 12048 & 30692 & 35193 & 18425 & 3630 & 110 \\ 110 & 3630 & 18425 & 35193 & 30692 & 12048 & 1677 & 33 \\ 130 & 3136 & 12303 & 18425 & 12048 & 3134 & 215 & 1 \\ 57 & 1027 & 3136 & 3630 & 1677 & 215 & 2 & 0 \\ 5 & 57 & 130 & 110 & 33 & 1 & 0 & 0 \end{bmatrix}.$$

The Box Spline $M_{2,2,2}$. The box spline $M_{2,2,2}$ has already been used in wavelet type decompositions (but not with orthogonal wavelets or pre-wavelets) by De Vore, Jawerth, and Lucier [DJL] for surface compression. It is a C^2 piecewise quartic that has support equal to $\{x : x = t_1(1,0) + t_2(0,1) + t_3(1,1),\ 0 \leqslant t_r \leqslant 2\}$ and is symmetric about the point $c = (2,2)$. The refinement mask, a_φ, for $\varphi = M_{2,2,2}$ on $[0..4]^2 \cap \mathbb{Z}^2 \supseteq \operatorname{supp} a_\varphi$ is

$$a_\varphi = \frac{1}{64} \begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 6 & 6 & 2 \\ 1 & 6 & 10 & 6 & 1 \\ 2 & 6 & 6 & 2 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{bmatrix}.$$

The support of the sequences $(M^c_{2,2,2} * M^c_{2,2,2})_{\mathrm{I}}$ and β_0 for this box spline are more substantial. The first has support inside $[-4..4]^2 \cap \mathbb{Z}^2$:

$$(M^c_{2,2,2} * M^c_{2,2,2})_{\mathrm{I}} =$$

$$\frac{1}{362880}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 34 & 34 & 2 & 0 \\ 0 & 0 & 0 & 34 & 1736 & 5100 & 1736 & 34 & 0 \\ 0 & 0 & 34 & 5100 & 37742 & 37742 & 5100 & 34 & 0 \\ 0 & 2 & 1736 & 37742 & 94992 & 37742 & 1736 & 2 & 0 \\ 0 & 34 & 5100 & 37742 & 37742 & 5100 & 34 & 0 & 0 \\ 0 & 34 & 1736 & 5100 & 1736 & 34 & 0 & 0 & 0 \\ 0 & 2 & 34 & 34 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then β_0 has supp $\beta_0 \subseteq [-3..7]^2 \cap \mathbb{Z}^2$ and on that square has values

$$\beta_0 = \frac{1}{23224320} \times$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 38 & 104 & 104 & 38 & 2 \\ 0 & 0 & 0 & 0 & 38 & 1884 & 8890 & 14088 & 8890 & 1884 & 38 \\ 0 & 0 & 0 & 104 & 8890 & 69054 & 163440 & 163440 & 69054 & 8890 & 104 \\ 0 & 0 & 104 & 14088 & 163440 & 555000 & 811088 & 555000 & 163440 & 14088 & 104 \\ 0 & 38 & 8890 & 163440 & 811088 & 1677952 & 1677952 & 811088 & 163440 & 8890 & 38 \\ 2 & 1884 & 69054 & 555000 & 1677952 & 2380248 & 1677952 & 555000 & 69054 & 1884 & 2 \\ 38 & 8890 & 163440 & 811088 & 1677952 & 1677952 & 811088 & 163400 & 8890 & 38 & 0 \\ 104 & 14088 & 163440 & 555000 & 811088 & 555000 & 163440 & 14088 & 104 & 0 & 0 \\ 104 & 8890 & 69054 & 163440 & 163440 & 69054 & 8890 & 104 & 0 & 0 & 0 \\ 38 & 1884 & 8890 & 14088 & 8890 & 1884 & 38 & 0 & 0 & 0 & 0 \\ 2 & 38 & 104 & 104 & 38 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is a relatively easy matter to change the sign pattern and translate the sequence β_0 to come up with β_μ and therefore the values of the pre-wavelets φ_μ. The pre-wavelets for $M_{2,2,2}$ are shown in Fig. 1. The smallest rectangle that contains the support of all the pre-wavelets is $[-2..6]^2$. The pre-wavelet $\varphi_{(1,0)}$ has center $(1, 1)$ and the smallest rectangle that contains its support is $[-2..5]^2$. The pre-wavelets $\varphi_{(0,1)}$ and $\varphi_{(0,1)}$ have centers $(\frac{3}{2}, 1)$ and $(1, \frac{3}{2})$ respectively, with $[-1.5..5.5] \times [-2..5]$ and $[-2..5] \times [-1.5..5.5]$ being the smallest rectangles that contain their respective supports. The fact that the coefficients β_μ come from multipliers

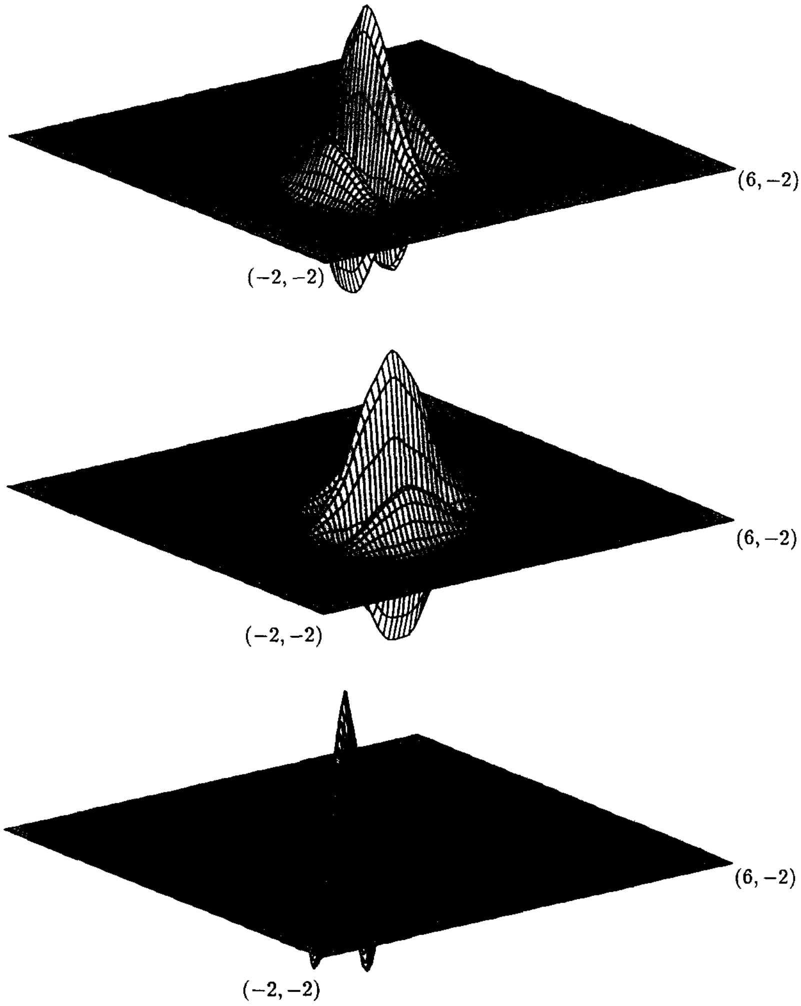

FIGURE 1. The prewavelets $\varphi_{(1,0)}$, $\varphi_{(0,1)}$ and $\varphi_{(1,1)}$ for $M_{2,2,2}$ on $[-2..6]^2$

that are rotations and translations of a single function is dramatically illustrated in Fig. 1.

REFERENCES

[B] G. BATTLE, A block spin construction of ondelettes. I. Lemarié functions, *Comm. Math. Phys.* **110** (1987), 601–615.

[CSW] C. K. CHUI, J. STÖCKLER, AND J. D. WARD, Compactly supported box spline wavelets, preprint.

[CW_1] C. K. CHUI AND J. Z. WANG, On compactly supported spline wavelets and a duality principle, *Trans. Amer. Math. Soc.*, to appear.

[CW_2] C. K. CHUI AND J. Z. WANG, A general framework of compactly supported splines and wavelets, *J. Approximation Theory*, to appear.

[D] I. DAUBECHIES, Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988), 909–996.

[DJL] R. DEVORE, B. JAWERTH, AND B. LUCIER, Surface compression, *CAGD*, to appear.

[JM_1] R. Q. JIA AND C. A. MICCHELLI, On linear independence of integer translates of a finite number of functions, *Proc. Roy. Soc. Edin., Series A*, to appear.

[JM_2] R. Q. JIA AND C. A. MICCHELLI, Using the refinement equations for the construction of pre-wavelets. II. Powers of two, *in* "Curves and Surfaces" (P. J. Laurent, A. LéMéhauté, and L. L. Schumaker, Eds.), Academic Press, New York, 1991.

[Ma] S. G. MALLAT, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$, *Trans. Amer. Math. Soc.* **315** (1989), 69–87.

[M] Y. MEYER, Ondelettes et fonctions splines, *in* "Seminaire Equations aux Dérivées Partielles," Ecole Polytechnique, Paris, France, 1986.

[Mi] C. A. MICCHELLI, Using the refinement equation for the construction of pre-wavelets, *Numer. Algorithms* **1** (1992), 75–116.

[RS] S. D. RIEMENSCHNEIDER AND Z. SHEN, Box splines, cardinal series and wavelets, *in* "Approximation Theory and Functional Analysis" (C. K. Chui, Ed.), pp. 133–149, Academic Press, New York, 1991.